Green-Haar method for fractional partial differential equations

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Abstract

Purpose – The purpose of this study is to obtain the numerical scheme of finding the numerical solutions of arbitrary order partial differential equations subject to the initial and boundary conditions.

Design/methodology/approach – The authors present a novel Green-Haar approach for the family of fractional partial differential equations. The method comprises a combination of Haar wavelet method with the Green function. To handle the nonlinear fractional partial differential equations the authors use Picard technique along with Green-Haar method.

Findings – The results for some numerical examples are documented in tabular and graphical form to elaborate on the efficiency and precision of the suggested method. The obtained results by proposed method are compared with the Haar wavelet method. The method is better than the conventional Haar wavelet method, for the tested problems, in terms of accuracy. Moreover, for the convergence of the proposed technique, inequality is derived in the context of error analysis.

Practical implications – The authors present numerical solutions for nonlinear Burger's partial differential equations and two-term partial differential equations.

Originality/value – Engineers and applied scientists may use the present method for solving fractional models appearing in applications.

Keywords Caputo derivative, Fractional partial differential equations, Green-Haar method, Haar wavelets, Picard technique

Paper type Research paper

1. Introduction

The fractional calculus and its applications in numerous fields of science and engineering are considered now an important field of mathematics capable to bring new incite in the dynamics of non-local complex systems (Hilfer, 2000; Kilbas *et al.*, 2006; Podlubny, 1999; Samko *et al.*, 1993). As the fractional calculus is a powerful apparatus to portray physical systems that have a long-term memory, which provides more options in the fields of mathematics and theoretical physics. Accordingly fractional differential equations have become famous for modeling many physical phenomena, such as in signal processing (Calderon *et al.*, 2006), anomalous diffusion (Cosenza and Korosak, 2014), visco-elasticity (Chen *et al.*, 2014), fluid dynamics (He, 1999), economics (Baillie, 1996), bio-engineering (Magin, 2006) and continuum and statistical mechanics (Carpinteri and Mainardi, 1997).

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Exact solutions of fractional partial differential equations seldom appear in the literary works; the demand to accomplishment few reliable and competent computational techniques is basic interest. To name a few in the literature, Abdulaziz *et al.* (2009) achieved estimated exact solutions for arbitrary real order Korteweg-de Vries equations along with the homotopy perturbation technique. Kurt and Tasbozan (2015) discussed homotopy analysis method on time-fractional Whitham–Broer–Kaup equation to obtain its approximate analytical solutions. A modified form of Adomian decomposition technique was developed by Odibat (2006) to solve fractional diffusion-wave problems. Non-polynomial quintic spline method used in Amin *et al.* (2019) for numerical solution of fourth-order time-fractional partial differential equations.

Latterly, wavelets got attention and attracted many researchers in the area of numerical approximations because of their extensive use. Wavelets theory is comparatively growing area in the mathematical application. A detailed survey of previous work can be found in Dahmen *et al.* (1997). Some special uses of wavelets in the context of computational technique comprise numerical integration and solutions of integral equations (Aziz and Siraj-ul-Islam, 2013; Siraj-ul-Islam and Haq, 2010), ordinary and partial differential equations (Comincioli *et al.*, 2000; Lepik and Hein, 2014), also fractional partial differential equations (Rehman and Khan, 2013; Saeed and Rehman, 2015; Wu, 2009). Numerous types of wavelets have been used in various implementations, for example, B-spline (Dehghan and Lakestani, 2008; Esen and Tasbozan, 2017), Legendre wavelets (Rehman and Khan, 2011), Haar wavelets (Lepik, 2007; Rehman and Khan, 2012), Daubechies (Diaz *et al.*, 2009), Battle–Lemarie (Zhu *et al.*, 1997) and Chebyshev (Babolian and Fattahzdeh, 2007). Haar wavelet have been used by a lot of scientists because of their clarity and good convergence rate. Mathematically, Haar wavelets family consists of rectangular functions. It also contains the lower member of the Daubechies family of wavelets, which is appropriate for numerical applications. Lepik (2011) and Celik (2012, 2013) introduced a numerical technique to obtain numerical solutions for linear and nonlinear partial differential equations, respectively, by using two-dimensional Haar wavelets. To test the method, Lepik used the diffusion besides Poisson equations whereas Celik used generalized Burgers–Huxley equation and magnetohydrodynamic flow equations.

In this work, a method depended on the two-dimensional Haar wavelet is proposed, called Green-Haar technique. This method extends the Green-Haar method developed in Rehman *et al.* (2019) for the numerical solutions of fractional ordinary differential equations. Green-Haar technique is used for solving fractional partial differential equations subject to the initial and boundary conditions. Here we concentrate on linear and nonlinear fractional partial differential equations. For certain problems this method does not require the use of operational matrices. However, for some class of problems Green-Haar method is used along with operational matrices. The convergence of the suggested method has been examined and elaborated in the context of error analysis. The suggested technique is suitable on account of solving such as linear/nonlinear fractional partial differential equations. To exhibit the effectiveness and precision of the suggested method, it is tested for some examples. The outcomes of these test problems are documented in the tabular and graphical form. Moreover, for comparison the results have also been documented in tabular form against previous studies.

This article is summarized as: in Section 2, we quoted a few necessary preliminaries, which help us in the upcoming sections. Section 3 focuses on Haar wavelets, which is the basic part of this study. Also, the function approximation and Haar wavelet operational matrices are explained. Green-Haar technique for the fractional partial differential equations and computational process of the technique for specific problems is discussed in Section 4. The convergence of the suggested technique is investigated in the context of error analysis in Section 5. Section 6 comprises results discussion of some applications to investigate the effectiveness of the suggested technique in graphical and tabular form. Also, we compared the outcomes of some tested problems with the previous studies. Finally, Section 7 contains a discussion and conclusion.

2. Preliminaries

For convenience, in this part, we introduce a few elementary concepts and definitions of fractional calculus. These preliminaries' facts are going to help in the upcoming sections.

Definition 2.1. Diethelm (2010) and Podlubny (1999). The partial Caputo fractional derivative of $u(x, t) \in C^n([0,1] \times [0,1])$ of order $\alpha \in \mathbb{R}^+$ with respect to x is defined as:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \begin{cases} \mathcal{I}_x^{n-\alpha} \dfrac{\partial^n u(x,t)}{\partial x^n}, & \alpha \in (n-1, n]; \\ \dfrac{\partial^n u(x,t)}{\partial x^n}, & \alpha = n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $n = \lfloor \alpha \rfloor + 1$ and $\mathcal{I}_x^{n-\alpha}$ are the Riemann–Liouville fractional integral, stated as:

$$\mathcal{I}_x^\alpha g(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x (x-\tau)^{\alpha-1} g(\tau) d\tau, & \alpha > 0; \\ g(x), & \alpha = 0. \end{cases} \tag{2.2}$$

Example 2.2. Consider $\alpha, \mu \in \mathbb{R}^+$ and $g(x) = x^\mu$. The Caputo fractional derivative of $g(x)$ is given as:

$$\mathcal{D}_x^\alpha g(x) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}. \tag{2.3}$$

For the function of one variable, we use the notation $\mathcal{D}_x^\alpha$ instead of notation $\frac{\partial^\alpha}{\partial x^\alpha}$.

We state a few elementary properties (Diethelm, 2010) of fractional integral and differential operators as:

- $\mathcal{I}_x^\alpha \mathcal{I}_x^\beta f(x) = \mathcal{I}_x^\beta \mathcal{I}_x^\alpha f(x) = \mathcal{I}_x^{\alpha+\beta} f(x)$.
- $\mathcal{I}_x^\alpha \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = u(x,t) - \sum_{i=0}^{n-1} \frac{x^i}{\Gamma(i+1)} \frac{\partial^i u(x,t)|_{x=0}}{\partial x^i}$.
- $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \mathcal{I}_x^\beta u(x,t) = \mathcal{I}_x^{\beta-\alpha} u(x,t)$.

Definition 2.3. Pang *et al.* (2018). The Mittag-Leffler function $\mathbb{E}_{\gamma,\beta}$ with dependence on two parameters α and β are stated as:

$$\mathbb{E}_{\gamma,\beta}(t) = \sum_{k=o}^{\infty} \frac{t^k}{\Gamma(\gamma k + \beta)}, \qquad \gamma, \beta > 0.$$

As a particular case, when $\beta = 1$ we have:

$$\mathbb{E}_{\gamma}(t) = \sum_{k=o}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)} = e^t, \qquad \gamma > 0.$$

Lemma 2.4. Diethelm (2010). *For* $\theta \in \mathbb{R}$, $\gamma > 0$ and $\beta > 1$, we have:

$$\mathcal{D}_t^{\alpha}(t-a)^{\beta-1}\mathbb{E}_{\gamma,\beta}\left[\theta(t-a)^{\gamma}\right] = (t-a)^{\beta-\alpha-1}\mathbb{E}_{\gamma,\beta-\alpha}\left[\theta(t-a)^{\gamma}\right], \tag{2.4}$$

and for $\beta = 1$ and $\alpha = \gamma$, we have:

$$\mathcal{D}_t^{\alpha}\mathbb{E}_{\alpha}\left[\theta(t-a)^{\alpha}\right] = \theta\,\mathbb{E}_{\alpha}\left[\theta(t-a)^{\alpha}\right]. \tag{2.5}$$

Lemma 2.5. Wang *et al.* (2014). Assume that the function $\frac{\partial u(x,t)}{\partial x}$ is continuous and bounded on (0, 1) × (0, 1) that is there exist M > 0 $|\frac{\partial u(x,t)}{\partial x}| \leq M \,\forall\, x, t \in (0, 1) \times (0, 1)$, and also assume that $u^m(x, t)$ obtained by using Haar wavelet, are the approximations of $u(x, t)$, then we have:

$$|u^m(x,t) - u(x,t)| \leq \frac{M}{\sqrt{3}}\frac{1}{m^3}.$$

3. Haar wavelets and function approximations

A class of wavelet $\{\psi_{j,i}(t)\}$ where $j \in \mathbb{R}$ and $i \in \mathbb{R}$ is an orthonormal subclass of the Hilbert space $\mathbb{L}_2(\mathbb{R})$. This function ψ is called mother wavelet, all function in the wavelet class are generated from ψ, which verify the relation given below:

$$\psi_{j,i}(t) = 2^{\frac{j}{2}}\psi(2^j t - i).$$

The Haar wavelet is step functions over the real line. These functions are restrained to the values: −1, 0 and 1. Each function that falls in category of Haar wavelets is essentially stated over $t \in [a, b]$ other than the scaling function conveyed in Lepik and Hein (2014) as:

$$h_i(t) = \begin{cases} 1, & \text{for } t \in \left[\xi_1(i), \xi_2(i)\right]; \\ -1, & \text{for } t \in \left[\xi_2(i), \xi_3(i)\right]; \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\xi_1(i) = a + (b-a)\frac{k}{m}$, $\xi_2(i) = a + (b-a)\frac{2k+1}{2m}$, $\xi_3(i) = a + (b-a)\frac{k+1}{m}$. We define the quantity $m = 2^j$, $j = 0, 1, 2, 3, \ldots, j$ and $k = 0, 1, 2, 3, \ldots, m-1$. Here the parameter j is used as a representation for the level of wavelet or dilation parameter, translation is represented by k, while the maximal level of resolution for the Haar wavelet is represented by j. The connection among the parameters m, k and i is given as $i = m + k + 1$.

The equation (3.1) is valid for $i \geq 3$. It is presumed that the value $i = 1$ and $i = 2$ corresponds to the following scaling function and mother wavelet, respectively:

$$h_1(t) = \begin{cases} 1, & \text{for } t \in [a,b]; \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

and

$$h_2(t) = \begin{cases} 1, \; t \in \left[a, \dfrac{a+b}{2}\right); \\ -1, \; t \in \left[\dfrac{a+b}{2}, b\right); \\ 0, \quad \text{otherwise.} \end{cases} \tag{3.3}$$

If $u(t) \in L^2[0, 1]$, it can be represented by Haar wavelets as:

$$u(t) = \sum_{i=0}^{\infty} c_i h_i(t), \tag{3.4}$$

where $c_i = \langle u(t), h_i(t) \rangle$. In particular, we can approximate $u(t)$ by truncated series:

$$u(t) \cong u_m(t) = \sum_{i=0}^{m-1} c_i h_i(t). \tag{3.5}$$

A function of two variables $u(x, t)$ can be estimated by Haar wavelet as:

$$u(x,t) \cong u_m(x,t) = \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} c_{j,i} h_j(x) h_i(t) = H^t(x) C H(t).$$

The operational matrix P^α for the fractional-order integration of the Haar wavelet is derived and discussed in Lepik and Hein (2014).

4. Green-Haar method and numerical procedure to fractional partial differential equations

In this section, we discuss Green-Haar technique for fractional partial differential equations with initial and boundary conditions and describes the numerical procedure, namely, Green-Haar technique to approximate the numerical solutions.

4.1 Green-Haar method

The fractional Green's function is defined, proposed and exploited by Miller and Ross (1993) applied to fractional differential equations consisting of derivatives of order $k\alpha$ only, where $k \in \mathbb{R}$. We propose a new technique to obtain the solutions of linear and nonlinear fractional partial differential equations numerically, called Green-Haar technique. In general, this method does not require to use the operational matrix for the fractional partial differential equation. However, for some cases Green-Haar is used along with operational matrix. The study undertaken reveal that the technique is even more computationally capable against some of the relevant numerical methods discussed in previous studies. Interestingly, accuracy is not compromised, rather enhanced by using Green-Haar method for solving fractional partial differential equations subject to the initial and boundary values.

4.2 Numerical procedure

We apply the Green-Haar technique to estimate the numerical solutions to the linear fractional partial differential equations with initial and boundary values. Particularly, we consider the following form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a(x)\frac{\partial^\beta u(x,t)}{\partial x^\beta} - b(x)\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} - d(x)u(x,t) + g(x,t), \tag{4.1}$$

with the initial and boundary conditions:

$$(i)\ \ u(x,0) = p_1(x),\ \ u(x,1) = p_2(x)\ \ or\ \ (ii)\ \ u(x,0) = p_1(x),\ \ \frac{\partial u(x,t)}{\partial t}|_{t=0} = q_1(x), \tag{4.2}$$

and

$$u(0,t) = r_1(t),\ \ u(1,t) = r_2(t), \tag{4.3}$$

where $1 < \alpha \le 2$, $1 < \beta \le 2$ and $0 < \gamma \le 1$. Firstly, we approximate the term involving derivative of order β by Haar wavelets as:

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = H^t(x)CH(t). \tag{4.4}$$

Applying the integral operator on both sides of equation (4.4), we have:

$$u(x,t) = \mathcal{I}_x^\beta H^t(x)CH(t) + x\Psi_1(t) + \Psi_2(t). \tag{4.5}$$

Using the conditions in equation (4.3), from equation (4.5), we obtain:

$$\Psi_2(t) = r_1(t),\quad \Psi_1(t) = -\left(\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}H^t(\tau)d\tau\right)CH(t) + r_2(t) - r_1(t).$$

Therefore, equation (4.5) will become:

$$\begin{aligned} u(x,t) &= \mathcal{I}_x^\beta H^t(x)CH(t) - x\left(\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}H^t(\tau)d\tau\right)CH(t) + x(r_2(t) - r_1(t)) + r_1(t), \\ &= \int_0^x \frac{(x-\tau)^{\beta-1}}{\Gamma(\beta)}H^t(\tau)d\tau CH(t) - x\left(\left(\int_0^x \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)} + \int_x^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}\right)\right. \\ &\quad H^t(\tau)d\tau)CH(t) + x(r_2(t) - r_1(t)) + r_1(t). \end{aligned} \tag{4.6}$$

Thus, we have:

$$u(x,t) = -\left(\int_0^1 G_1(x,\tau)H^t(\tau)d\tau\right)CH(t) + x(r_2(t) - r_1(t)) + r_1(t), \tag{4.7}$$

the function $G_1(x, \tau)$ in equation (4.7) given by:

$$G_1(x,\tau) = \begin{cases} \frac{1}{\Gamma(\beta)}\left(-(x-\tau)^{\beta-1} + x(1-\tau)^{\beta-1}\right), & \text{if } 0 \leq \tau < x; \\ \frac{x}{\Gamma(\beta)}(1-\tau)^{\beta-1}, & \text{if } x \leq \tau \leq 1, \end{cases} \tag{4.8}$$

is called the Green's function for boundary value problem equations (4.4) and (4.2[i]). The graph for the function $G_1(x, \tau)$, for $\alpha = 2$ and $j = 5$, is shown in Figure 1.

We approximate this Green's function by Haar wavelets as:

$$G_1(x,\tau) = H^t(x)\hat{G}_1^t H(\tau). \tag{4.9}$$

Equation (4.7) becomes:

$$u(x,t) = -H^t(x)\hat{G}_1^t CH(t) + x(r_2(t) - r_1(t)) + r_1(t). \tag{4.10}$$

By using the orthogonality (Babolian and Shahsavaran, 2009) of the sequence $\{h_i(t)\}$ on [0, 1], we have:

$$\int_0^1 H(\tau)H^t(\tau)d\tau = \mathbb{I}_{m\times m}, \tag{4.11}$$

where $\mathbb{I}_{m\times m}$ is an identity matrix of order m. Applying $\frac{\partial^\gamma}{\partial x^\gamma}$ on equation (4.6), we get:

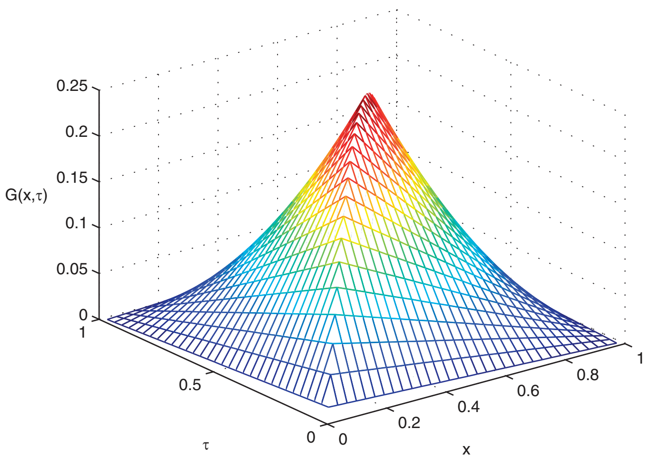

Figure 1. Green function for fixed values of $j = 5$ and for $\alpha = 2$

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} = \mathcal{I}^{\beta-\gamma}H^t(x)CH(t) - \left(\frac{x^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}H^t(\tau)d\tau\right)CH(t) + \frac{x^{1-\gamma}}{\Gamma(2-\gamma)}(r_2(t)-r_1(t)),$$
$$= \left(\int_0^1 G_2(x,\tau)H(\tau)d\tau\right)CH(t) + \frac{x^{1-\gamma}}{\Gamma(2-\gamma)}(r_2(t)-r_1(t)), \tag{4.12}$$

where

$$G_2(x,\tau) = \begin{cases} \frac{1}{\Gamma(\beta-\gamma)}(x-\tau)^{\beta-\gamma-1} - \frac{x^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(\beta)}(1-\tau)^{\beta-1}, & \text{if } 0 \le \tau < x; \\ -\frac{x^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(\beta)}(1-\tau)^{\beta-1}, & \text{if } x \le \tau \le 1. \end{cases}$$

Similarly, we have:

$$G_2(x,\tau) = H^t(x)\hat{G}_2^t H(\tau). \tag{4.13}$$

Now, the equation (4.12) becomes:

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} = H^t(x)\hat{G}_2^t CH(t) + \frac{x^{1-\gamma}}{\Gamma(2-\gamma)}(r_2(t)-r_1(t)). \tag{4.14}$$

Putting equations (4.4), (4.10) and (4.14) into equation (4.1), we obtain:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a(x)H^t(x)CH(t) - b(x)H^t(x)\hat{G}_2^t CH(t) + d(x)H^t(x)\hat{G}_1^t CH(t) - \frac{b(x)x^{1-\gamma}}{\Gamma(2-\gamma)}(r_2(t)-r_1(t)) - xd(x)(r_2(t)-r_1(t)) - d(x)r_1(t) + g(x,t). \tag{4.15}$$

For simplification, we use some convenient notations:

$$s(x,t) = -\frac{b(x)x^{1-\gamma}}{\Gamma(2-\gamma)}(r_2(t)-r_1(t)) - xd(x)(r_2(t)-r_1(t)) - d(x)r_1(t) + g(x,t), \quad y(x,t) = -x(r_2(t)-r_1(t)) - r_1(t) + t(p_2(x)-p_1(x)) + p_1(x)$$

and $z(x,t) = -x(r_2(t)-r_1(t)) - r_1(t) + tp_1(x) + q_1(x)$. Equation (4.15) becomes:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a(x)H^t(x)CH(t) - b(x)H^t(x)\hat{G}_2^t CH(t) + d(x)H^t(x)\hat{G}_1^t CH(t) + s(x,t). \tag{4.16}$$

Now, applying $\mathcal{I}_t^\alpha$ on both sides of equation (4.16), we have an alternative term for the function $u(x, t)$. Here it is difficult to calculate the fractional integral $\mathcal{I}_t^\alpha s(x,t)$ or even hard to calculate instantly. To overcome this difficulty, we approximate the function $s(x, t)$ by two-dimensional Haar wavelets form, as $s(x,t) = H^t(x)\hat{S}^t H(t)$ and then apply the fractional integral, we get:

$$u(x,t) = \left\{\left[a(x)H^t(x) - b(x)H^t(x)\hat{G}_2^t + d(x)H^t(x)\hat{G}_1^t\right]C + H^t(x)\hat{S}^t\right\}\mathcal{I}_t^\alpha H^t(t) + t\phi_1(x) + \phi_2(x). \tag{4.17}$$

By using the initial conditions equation (4.2[i]), from equation (4.17), we have $\phi_2(x) = p_1(x)$ and $\phi_1(x) = -\left\{\left[a(x)H^t(x) - b(x)H^t(x)\hat{G}_2^t + d(x)H^t(x)\hat{G}_1^t\right]C + H^t(x)\hat{S}^t\right\}\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}H^t(\tau)d\tau + p_2(x) - p_1(x)$. Therefore, equation (4.17) becomes:

$$u(x,t) = \left\{\left[a(x)H^t(x) - b(x)H^t(x)\hat{G}_2^t + d(x)H^t(x)\hat{G}_1^t\right]C + H^t(x)\hat{S}^t\right\}\int_0^1 G_3(t,\tau)H(\tau)d\tau + t(p_2(x) - p_1(x)) + p_1(x), \tag{4.18}$$

where

$$G_3(t,\tau) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left((t-\tau)^{\alpha-1} - t(1-\tau)^{\alpha-1}\right), & \text{if } 0 \le \tau < t; \\ -\frac{t}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}, & \text{if } t \le \tau \le 1, \end{cases}$$

and

$$G_3(t,\tau) = H^t(t)\hat{G}_3^t H(\tau) = H^t(\tau)\hat{G}_3^t H(t). \tag{4.19}$$

Equation (4.18) implies:

$$u(x,t) = \left\{\left[a(x)H^t(x) - b(x)H^t(x)\hat{G}_2^t + d(x)H^t(x)\hat{G}_1^t\right]C + H^t(x)\hat{S}^t\right\}\hat{G}_3^t H(t) + t(p_2(x) - p_1(x)) + p_1(x). \tag{4.20}$$

Now, by using initial conditions equation (4.2[ii]), we obtained $\phi_2(x) = p_1(x)$, $\phi_1(x) = q_1(x)$. Therefore, equation (4.17) becomes:

$$u(x,t) = \left\{\left[a(x)H^t(x) - b(x)H^t(x)\hat{G}_2^t + d(x)H^t(x)\hat{G}_1^t\right]C + H^t(x)\hat{S}^t\right\}P^\alpha + tq_1(x) + p_1(x). \tag{4.21}$$

Let A, B and D be the diagonal matrices, which are defined as follow:

$$A = \begin{bmatrix} a(x_1) & 0 & \cdots & 0 \\ 0 & a(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a(x_m) \end{bmatrix}, B = \begin{bmatrix} b(x_1) & 0 & \cdots & 0 \\ 0 & b(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b(x_m) \end{bmatrix},$$

$$D = \begin{bmatrix} d(x_1) & 0 & \cdots & 0 \\ 0 & d(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(x_m) \end{bmatrix},$$

where $x_i = \frac{2i-1}{2m}$ for $i = 1, 2, \ldots, m$.

From equations (4.10) and (4.20), we will have the following matrix form:

$$\hat{G}_1^t C - [H^t]^{-1}\left\{\left[AH^t - BH^t\hat{G}_2^t + DH^t\hat{G}_1^t\right]C\right\}\hat{G}_3^t = S^t\hat{G}_3^t + [H^t]^{-1}Y^t[H]^{-1}, \tag{4.22}$$

where Y is the approximation of the function $y(x, y) = -x(r_2(t) - r_1(t)) - r_1(t) + t(p_2(x) - p_1(x)) + p_1(x)$. The equation (4.22) is called Sylvester equation. To find the value of C we have to solve the algebraic system equation (4.22), and putting the value of C into equation (4.10) or in equation (4.21) to get the approximate solution at the collocation points. Also, from equations (4.10) and (4.21), we have the following algebraic system:

$$\hat{G}_1^t C - [H^t]^{-1}\left\{\left[AH^t - BH^t\hat{G}_2^t + DH^t\hat{G}_1^t\right]C\right\}P^\alpha = S^tP^\alpha + [H^t]^{-1}Z^t[H]^{-1}. \tag{4.23}$$

5. Convergence analysis

In this part, we derive inequality in the context of upper bound, which shows the convergence of Green-Haar technique for fractional partial differential equations. The convergence of Haar wavelet for partial differential equations is discussed in Wang *et al.* (2014). Proceeding in the same way, we extend the analysis for the present proposed technique. Here we use the notation $\mathcal{D}_x^\alpha$ instead of $\frac{\partial^\alpha}{\partial x^\alpha}$.

Theorem 5.1. Suppose that the function $\frac{\partial u_{k+1}(x,t)}{\partial x}$ is continuous and bounded on $(0, 1) \times (0, 1)$ that is there exist $M > 0$, such that $\left|\frac{\partial u_{k+1}(x,t)}{\partial x}\right| \leq M \forall x, t \in (0,1) \times (0,1)$, and also assume that $u_{k+1}^m(x,t)$ is an approximation of $u_{k+1}(x, t)$, then we have:

$$|u_{k+1}^m(x,t) - u_{k+1}(x,t)| \le \frac{M}{\sqrt{3}\Gamma(\alpha+1)} \frac{1}{m^3}.$$

Proof. Consider:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = u(x,t) + f(u(x,t),x,t), \qquad x,t \in [0,1], \tag{5.1}$$

with boundary conditions $u(0, t) = u_0(t)$, $u(1, t) = u_1(t)$. Now, applying the Picard technique to equation (5.1), we have:

$$\frac{\partial^\alpha u_{k+1}(x,t)}{\partial x^\alpha} = u_{k+1}(x,t) + f(u_k(x,t),x,t), \qquad x,t \in [0,1], \tag{5.2}$$

with boundary conditions $u_{k+1}(0, t) = u_0(t)$, $u_{k+1}(1, t) = u_1(t)$. The integral representation of equation (5.2) after using boundary conditions can be written as:

$$u_{k+1}(x,t) = \int_0^1 G(x,\xi) u_{k+1}(\xi,t) d\xi + h(x,t), \tag{5.3}$$

where $h(x,t) = \mathcal{I}_x^\alpha f(u_k(x,t),x,t) - x\mathcal{I}_x^\alpha f(u_k(1,t),1,t) + (u_1(t) - u_0(t))x + u_0(t)$. The functions $h(x, t)$ and $u_k(x, t)$ are known and can be used to obtain $u_{k+1}(x, t)$. Let $u_{k+1}^m(x,t)$ is the approximation of $u_{k+1}(x, t)$, then we have:

$$|u_{k+1}^m(x,t) - u_{k+1}(x,t)| \le \int_0^1 G(x,\xi) |u_{k+1}^m(\xi,t) - u_{k+1}(\xi,t)| d\xi. \tag{5.4}$$

Therefore, from Lemma 2.5, we have:

$$|u_{k+1}^m(x,t) - u_{k+1}(x,t)| \le \frac{M}{\sqrt{3}} \frac{1}{m^3} \int_0^1 G(x,\xi) d\xi. \tag{5.5}$$

We evaluate the integral $\int_0^1 G(x,\xi) d\xi$ as:

$$\int_0^1 G(x,\xi) d\xi = \frac{1}{\Gamma(\alpha)} \left\{ \int_0^x \left[(x-\xi)^{\alpha-1} - x(1-\xi)^{\alpha-1} \right] d\xi - x\int_x^1 (1-\xi)^{\alpha-1} d\xi \right\},$$

$$= \frac{1}{\Gamma(\alpha+1)} (x^\alpha - x).$$

As $-x \le 0$ because $x \in [0, 1]$, therefore:

$$\int_0^1 G(x,\xi) d\xi \le \frac{x^\alpha}{\Gamma(\alpha+1)} \le \frac{1}{\Gamma(\alpha+1)}. \tag{5.6}$$

Using equation (5.6) into equation (5.5), we obtain:

$$|u_{k+1}^m(x,t) - u_{k+1}(x,t)| \leq \frac{M}{\sqrt{3}\Gamma(\alpha+1)} \frac{1}{m^3}. \tag{5.7}$$

6. Applications

We apply Green-Haar method to obtain the numerical solution of the fractional partial differential equations subject to the initial and boundary conditions. We also compare the outcome of the tested problem with previous studies.

Example 6.1. Consider:

$$\frac{\partial u(x,t)}{\partial t^\alpha} = \frac{x^2}{2} \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad x \in [0,1] \text{ and } t \in \mathbb{R}, \tag{6.1}$$

with the initial and boundary conditions:

$$u(x,0) = x, \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = x^2, \tag{6.2}$$

$$u(0,t) = 0, \quad u(1,t) = 1 + tE_{\alpha,2}(t^\alpha), \tag{6.3}$$

where $1 < \alpha \leq 2$. The series solution of the equation (6.1) is $u(x,t) = x + x^2 \sum_{k=0}^{\infty} \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}$. The numerical solutions can be obtained by the method discussed in Section 4.2. The absolute error for $\alpha = 1.75$ and at different values of j are shown in Table I. We also compare the proposed method with Haar wavelet method discussed in Rehman and Khan (2013) by using L_∞ and L_2 norms. The absolute error obtained for proposed method is slightly better than the Haar wavelet method. We observe that the maximum absolute error decrease by increasing value of j.

Example 6.2. Consider the fractional convection-diffusion equation (6.4):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = p(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} - q(x) \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + f(x,t), \; 1 < \alpha \leq 2, \; 1 < \beta \leq 2, \text{ and } 0 < \gamma \leq 1, \tag{6.4}$$

subject to the initial and boundary conditions $u(x,0) = u(x,1) = 0$, $u(x,t) = u(1,t) = 0$,

Table I. Maximum absolute error for $\alpha = 1.75$, $t = 0.2$ and for different values of j

j	Haar wavelet method Rehman and Khan (2013) L_∞	L_2	Green-Haar Method L_∞	L_2
03	3.78034×10^{-4}	6.23867×10^{-4}	6.50728×10^{-5}	1.21168×10^{-4}
04	9.55847×10^{-5}	2.22552×10^{-4}	2.03649×10^{-5}	4.68900×10^{-5}
05	2.45589×10^{-5}	7.92660×10^{-5}	5.14014×10^{-6}	1.62794×10^{-5}
06	6.13517×10^{-6}	2.81917×10^{-5}	1.29440×10^{-6}	5.73428×10^{-6}
07	1.78149×10^{-6}	9.93583×10^{-6}	3.24483×10^{-7}	2.02171×10^{-6}

where $p(x) = \Gamma(\alpha\beta - \alpha + 2)\Gamma(b - 2\alpha + 1)x^{\alpha}$, $q(x) = \Gamma(\alpha\beta + 2 - \beta)\Gamma(b - \alpha - \beta + 1)x\beta$ and $f(x,t) = -(x^{a\beta} - x^{b-\alpha})(d\pi)^3 t^{3-\alpha} E_{2,4-\alpha}\left(-(d\pi t)^2\right) + \{\Gamma(a\beta+2)[\Gamma(b-\alpha-\beta+1) - \Gamma(b-2\alpha+1)]x^{a\beta+1} + \Gamma(b+\alpha+1)[\Gamma(a\beta-\alpha+2) - \Gamma(a\beta+2-\beta)]x^{b-\alpha}\}\sin(d\pi t)$. Therefore, it can be conventionally verified that the exact solution to equation (6.4) is $u(x, t) = (x^{\alpha\beta+1} - x^{b-a})\sin(d\pi t)$. Numerical solutions are obtain for $a = 2$, $b = d = 4$, $\alpha = 2$, $\gamma = 0.75$, $\beta = 2$ and different values of j are shown in the tabular form in the Table II. The numerical and exact solutions for $j = 6$, $\alpha = 2$, $\beta = 2$ and $\gamma = 0.35$ are plotted in the Figure 4. Also, the numerical and exact solution and their absolute error are plotted in the graphical form in the Figures 2 and 3, respectively, for fixed values of $j = 6$, $\alpha = 1.85$, $\beta = 2$ and $\gamma = 0.75$.

6.1 Nonlinear case

The nonlinear partial differential equations are transformed into linear form by using Picard technique and then solved numerically. We applying the Picard method for the nonlinear fractional partial differential equation to get discretized form. Consequently, Green-Haar technique is used for discretized form to get the numerical solutions. Consider:

t	x	$j = 5$	$j = 6$	$j = 7$	Exact solution for $j = 7$
0.2	0.25	−0.0184739	−0.0183575	−0.0183750	−0.0183611
	0.50	−0.0432248	−0.0430080	−0.0430584	−0.0430186
	0.75	−0.0444660	−0.0443182	−0.0442666	−0.0442718
0.4	0.25	0.0296512	0.0297685	0.0296136	0.0296912
	0.50	0.0693773	0.0697418	0.0696283	0.0695640
	0.75	0.0713694	0.0717829	0.0716655	0.0715904
0.6	0.25	−0.0297712	−0.0297685	−0.0297136	−0.0296912
	0.50	−0.0693773	−0.0697418	−0.0696283	−0.0695640
	0.75	−0.0713694	−0.0717829	−0.0716655	−0.0715904

Table II. Exact and numerical solution for $\alpha = 2$, $\gamma = 0.75$, $\beta = 2$, $a = 2$, $b = d = 4$ and for different values of j

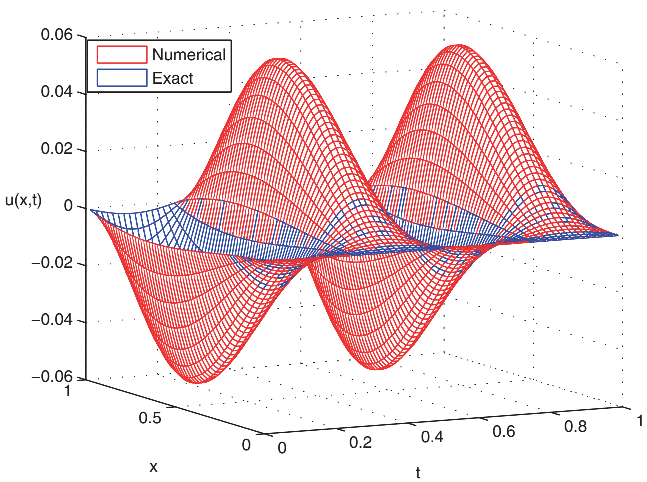

Figure 2. Exact and numerical solution for $j = 6$, $\alpha = 1.85$, $\beta = 2$ and $\gamma = 0.75$

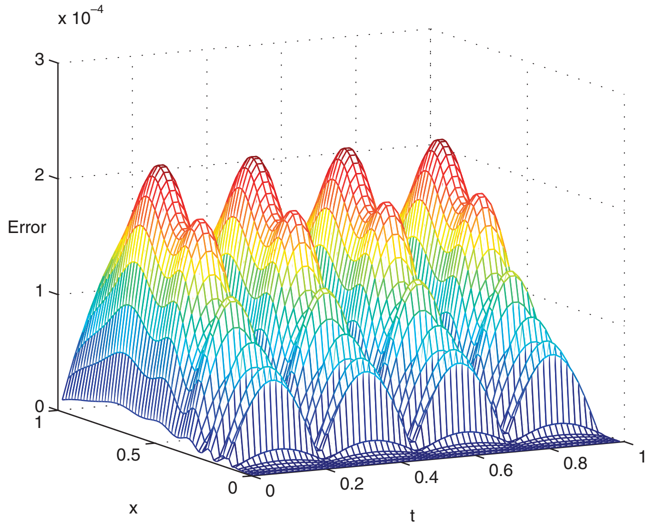

Figure 3.
Absolute error for $j = 6$, $\alpha = 1.85$, $\beta = 2$ and $\gamma = 0.75$

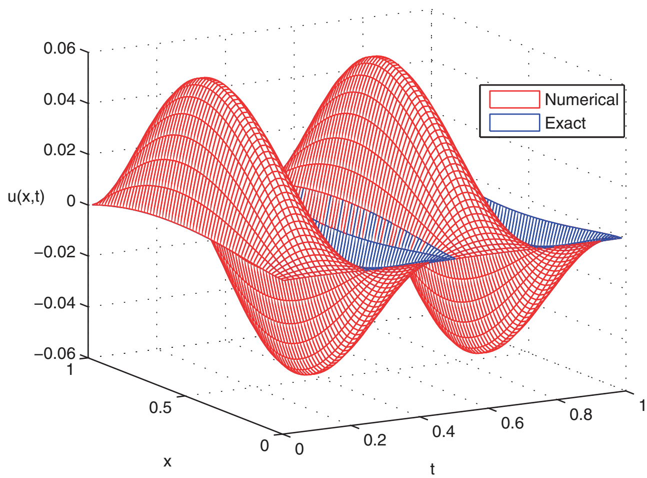

Figure 4.
Exact and numerical solutions for the fixed values of $j = 6$, $\alpha = 2$, $\beta = 2$ and $\gamma = 0.35$

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = p(x)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + q(x)u(x,t)\frac{\partial^{\gamma} u(x,t)}{\partial^{\gamma} x^{\gamma}} + r(x)u^{n}(x,t) + g(x,t), \tag{6.5}$$

where $1 < \alpha \leq 2, 1 < \beta \leq 2, 0 < \gamma \leq 1, n > 1$, also with the initial and boundary conditions in equations (4.2) and (4.3).

Using the Picard iteration (Bellman and Kalaba, 1965) to equation (6.5), we obtain:

$$\frac{\partial^{\alpha} u_{k+1}(x,t)}{\partial t^{\alpha}} - p(x)\frac{\partial^{\beta} u_{k+1}(x,t)}{\partial x^{\beta}} = y\left(x, t, u_k(x,t), \frac{\partial^{\gamma} u_k(x,t)}{\partial x^{\gamma}}\right), \tag{6.6}$$

subject to the initial and boundary conditions:

$$(i)\ \ u_{k+1}(x,0)=a(x),\ \ u_{k+1}(x,1)=b(x)\ \ or\ \ (ii)\ \ u_{k+1}(x,0)=c(x),\ \ \frac{\partial u_{k+1}(x,t)}{\partial t}|_{t=0}=d(x), \quad (6.7)$$

$$u_{k+1}(0,t)=f(t),\ \ u_{k+1}(1,t)=h(t). \quad (6.8)$$

where $y\left(x,t,u_k(x,t),\frac{\partial^\gamma u_k(x,t)}{\partial x^\gamma}\right)=q(x)u_k(x,t)\frac{\partial^\gamma u_k(x,t)}{\partial x^\gamma}+r(x)u_k^n(x,t)+g(x,t)$. Applying the Green-Haar method, which is discussed in Section 4 to get the numerical solutions.

Example 6.3. Consider the following Burger's equation with initial and boundary conditions:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+u(x,t)\frac{\partial u(x,t)}{\partial x}=\frac{\partial^\beta u(x,t)}{\partial x^\beta}\quad 0<\alpha\leq 1,\ \ 1<\beta\leq 2, \quad (6.9)$$

$$u(x,0)=2x,$$

$$u(0,t)=0,\ \ u(1,t)=\frac{2}{1+2t},$$

where $0\leq x\leq 1$ and $t\geq 0$. When $\alpha=1$ and $\beta=2$, the exact solution of the equation (6.9) is $u(x,t)=\frac{2x}{1+2t}$. Consider $u_0(x,t)=2x$ and $\frac{\partial u_0(x,t)}{\partial x}=2$ as an initial approximation and apply the Picard iteration technique to equation (6.9) to convert into discretized from. Then, Green-Haar method is applied for the numerical solutions. We plot the exact and numerical solution and the absolute error between exact and numerical solution by fixing the values of $\alpha=1$, $\beta=2$ and $j=5$ in Figure 5.

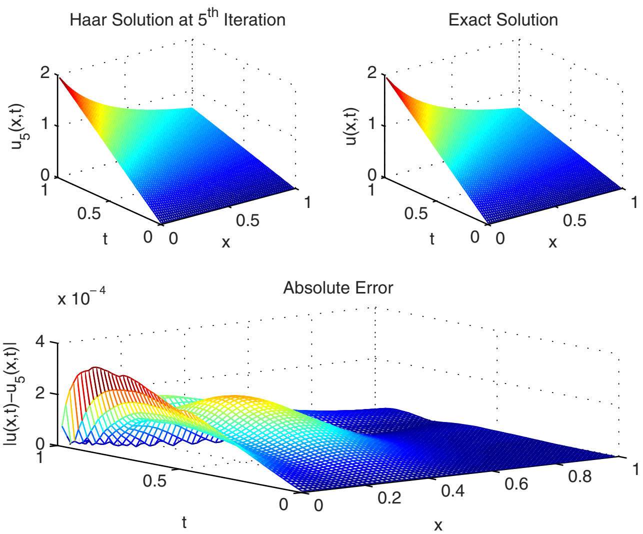

Figure 5.
Exact and numerical solutions and their absolute error for the fixed values of $j=5$, $\alpha=2$ and $\beta=1$

7. Conclusion

In this paper, Green-Haar technique is extended for the numerical solutions of linear and nonlinear fractional partial differential equations subject to the initial and boundary conditions. Picard technique is applied to transform nonlinear equations into the corresponding linear equation and then Green-Haar method is applied. The convergence of the suggested technique has also been documented in the context of error analysis. Some numerical examples are tested to check the effectiveness of the suggested technique. The outcomes of numerical examples are documented in the graphical and tabular form. According to Table I, Green-Haar method gives slightly more accurate results in comparison with the Haar wavelet technique. It shows that results obtained from Green-Haar method are in good agreement with exact solution when applied to the linear and nonlinear fractional partial differential equations.

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